

Connection between the Rayleigh and the Schrödinger equations

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We make a connection between the Schrödinger equation $D^2\Psi + (E - V)\Psi = 0$ and the Rayleigh equation $D(\rho DW) + (k^2/\Gamma^2)WD\rho - k^2\rho W = 0$ which is used to study the Rayleigh-Taylor instability of fluids in a gravitational field. Here D is the differential operator d/dy , $\rho(y)$ is the density profile of the fluid, $W(y)$ is the perturbed fluid velocity, k is the wave number of the perturbation, and $\Gamma^2 = \gamma^2/g$, where γ is the growth rate of the instability and g is the strength of the gravitational field. The connection between the Rayleigh and the Schrödinger equations is made by defining a potential $V(y)$ associated with $\rho(y)$, a wave function $\Psi(y)$ associated with $W(y)$, and an energy E associated with k . We consider several examples of the Rayleigh equation and show that they correspond to well-known problems in quantum mechanics such as a particle in a box, the harmonic oscillator, the Coulomb potential, etc. We illustrate the inversion symmetry of the Rayleigh equation under $\rho(y) \rightarrow 1/\rho(-y)$, and in an appendix we give and illustrate the more general potential $V(y)$, which includes surface tension and shear flow, the latter associated with the Kelvin-Helmholtz instability.

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I. INTRODUCTION

In this paper we make a connection between two equations, the Rayleigh equation and the Schrödinger equation, used in the two different fields of fluid dynamics (FD) and quantum mechanics (QM), respectively. Historically the Rayleigh equation was derived first [1]:

$$D(\rho DW) + \frac{k^2}{\Gamma^2} WD\rho - k^2\rho W = 0. \quad (1)$$

In this equation $\rho(y)$ is the density profile, $W(y)$ is the y component of the perturbed fluid velocity throughout the fluid, $k^2 = k_x^2 + k_z^2$, where k_x and k_z are the wave numbers of density perturbations which have the form $\delta\rho(y, t)e^{ik_x x + ik_z z}$, $\Gamma^2 = \gamma^2/g$, where g is the constant acceleration in the $+y$ direction (or, equivalently, a gravitational field in the $-y$ direction), and γ is the growth rate of the perturbations, i.e., $\delta\rho(y, t) = \delta\rho(y)e^{\gamma t}$. Finally, D stands for the operator d/dy . We follow Chandrasekhar's notation [2], where Eq. (1) is derived from the fluid equations. It applies for small or linear perturbations, i.e., $\delta\rho(y) \ll \rho(y)$ (see Refs. [1] and [2]).

The Schrödinger equation was proposed some 43 years later and reads [3]

$$D^2\Psi + (E - V)\Psi = 0, \quad (2)$$

where $V(y)$ is the potential, $\Psi(y)$ is the wave function, and E is the energy of the state described by Ψ . We are using standard notation where E and V are in units of $\hbar^2/2m$.

Equations (1) and (2) are second-order differential eigenvalue equations. In general there are many solutions (W_n, Γ_n) or (Ψ_n, E_n) , $n = 1, 2, 3, \dots$, which must be found by solving Eq. (1) or Eq. (2) subject to appropriate continuity and boundary conditions. Loosely speaking, $\rho(y) \sim V(y)$ and $W(y) \sim \Psi(y)$, while $E \sim k^2$. An exact relationship will be given below. Note that the Rayleigh equation has one free

variable, viz., k^2 , meaning that one must find the growth rates as functions of the wave number which varies continuously, $0 \leq k < \infty$.

Our interest stems primarily from the Rayleigh-Taylor (RT) instability [1,4] which can reduce the thermonuclear yield of inertial-confinement-fusion (ICF) capsules [5]. Perturbations will grow exponentially in time if Γ^2 is positive, and will oscillate in time if Γ^2 is negative since γ will be imaginary. In the latter case one has innocuous "gravity waves," thus the connection between Eqs. (1) and (2) gives another dimension to the familiar term "wave mechanics" applied to QM. Clearly, we are considering quite different "waves" here.

In addition to the physical problem of instabilities in ICF capsules the Rayleigh equation is widely used to study buoyancy problems in a fluid where the density varies continuously and to study the evolution of internal waves in oceans as well as the atmosphere. This wide applicability of the Rayleigh equation stems from the fact that it is the linearized limit of the Euler equation in a constant gravitational field. We do not include viscosity, i.e., the full Navier-Stokes equation, because its linearized version leads to a fourth-order differential equation (see [2]) which cannot be put in correspondence with the second-order Schrödinger equation.

Two benefits are expected to emerge from this cross fertilization of fields which, needless to say, is purely mathematical. First, we expect to find alternate analytic solutions to Eq. (1) or, less likely, to Eq. (2). The reason is that many more analytic solutions to the Schrödinger equation are known, compared with the one or two known for the Rayleigh equation. Second, we expect to contribute to the area of "potential recovery," i.e., the inverse problem of determining the potential $V(y)$ from the scattering phase shifts δ_n or the energy eigenvalues E_n . This is a well-known problem (the best example is the Pöschl-Teller potential [6]) and the process is known to lead to nonunique answers in QM. We hope to shed some light on this problem by applying the inversion theorem [7] to the Rayleigh equation and then making the connection to the Schrödinger equation. Accord-

ing to this theorem the two profiles $\rho(y)$ and $1/\rho(-y)$ have the same spectrum, i.e., the same set of eigenvalues (but not eigenfunctions) for all k . This is a nontrivial property whose physical origin is unknown. For example, for the discrete density profile $\rho=(1,3,10,6,2,30)$ we immediately conclude that a second profile, $(1,15,5,3,10,30)$ exists with the same “growth” rates for all k (“growth” in quotes because a few of the Γ^2 are negative and hence the associated eigenmodes are oscillatory in time). Whether there exists a third profile with the same spectrum is not known—our conjecture is that it does not [7].

As far as we know there is no similar or corresponding theorem for the QM case—given an arbitrary $V(y)$ one can calculate the spectrum; however, one cannot produce in a straightforward manner a second potential with exactly the same spectrum. The key words here are “arbitrary” and “straightforward manner”—otherwise one can find specific examples of phase equivalent potentials [6,8].

In Sec. II we give the general relations between the Rayleigh equation and the Schrödinger equation. Several examples are given in Sec. III illustrating this relationship. The inversion theorem is discussed in Sec. IV, and conclusions given in Sec. V. In the Appendix we extend the Rayleigh equation to include surface tension and shear flow, and discuss the potentials associated with them.

II. GENERAL RELATIONS

Equations (1) and (2) are linear in Ψ , W , and ρ , hence an overall factor in any of these three functions is immaterial. Define

$$q(y) \equiv \sqrt{\rho(y)}; \quad (3)$$

then it is straightforward to show that Eqs. (1) and (2) are equivalent if

$$\Psi = qW, \quad (4a)$$

$$V = \frac{D^2q}{q} - 2 \frac{k^2}{\Gamma^2} \frac{Dq}{q}, \quad (4b)$$

and

$$E = -k^2. \quad (4c)$$

One can always add an arbitrary constant to V and to E leaving $E - V$ unchanged, so Eq. (4c) does not limit us to negative-energy solutions. This will be clear by examples given later.

As mentioned in the Introduction, k is a real wave number. Equation (4c) suggests that it might be possible to define a new class of FD problems with imaginary k , a fairly common situation in QM where potentials often admit negative as well as positive-energy eigenstates. We will not pursue this somewhat intriguing possibility here, and confine ourselves to the traditional Rayleigh-Taylor instability with real k .

Given a $q(y)$, it is easy to construct the corresponding $V(y)$ via Eq. (4b). To go the opposite way is much more difficult since one must solve the equation

$$D^2q - 2 \frac{k^2}{\Gamma^2} Dq - Vq = 0 \quad (5)$$

to find $q(y)$ for a given $V(y)$.

A typical boundary condition in FD is a semi-infinite fluid of constant density. The corresponding QM condition is a constant V . In this constant density or constant V region $W(y)$ [and hence $\Psi(y)$ since $q = \text{const}$] decays exponentially with y , i.e., $\Psi \sim W \sim e^{-k|y|}$. A special case is a fixed boundary representing a wall: here $W = 0$. Similarly, $\Psi = 0$. A free boundary, on the other hand, admits a nonzero value of W , though it must obey another condition (see below). The fixed and free boundaries can be formally represented by $\rho = \infty$ and $\rho = 0$, which are related by inversion, and the inversion theorem relates the spectrum of the fixed case with that of the free case [7].

Unexpectedly, the most severe limitation in making the FD \leftrightarrow QM connection comes from the continuity requirements which are based on physical grounds: Ψ and W must be continuous. The limitation comes from Eq. (4a): if Ψ and W are to be continuous then q and hence ρ must also be continuous. This implies that FD problems with discontinuous density profiles cannot be mapped to a QM problem. In particular, the classical Rayleigh-Taylor problem which involves a sudden density jump cannot correspond to a QM problem. Rather than give up continuity of Ψ or W we will confine ourselves to continuous density profiles. In other words, defining a jump in a quantity f by $\Delta f = f_+ - f_-$, we limit ourselves to problems having $\Delta(\rho) = 0$.

Unlike the continuity of Ψ and W , which is dictated by physical requirements and hence beyond our control, there are no “outside” requirements on their derivatives, $D\Psi$ and DW , other than consistency with the Schrödinger equation and the Rayleigh equation, respectively. Integrating Eq. (1) over an infinitesimal distance we obtain [2]

$$\Delta(\rho DW) + \frac{k^2}{\Gamma^2} W \Delta(\rho) = 0. \quad (6)$$

Since we have already decided to limit ourselves to $\Delta(\rho) = 0$ problems, Eq. (6) implies that DW must also be continuous. [Equation (6) is modified if surface tension is present and then DW need not be continuous.]

At a free boundary the perturbation must satisfy the relation

$$DW + \frac{k^2}{\Gamma^2} W = 0, \quad (7)$$

which follows from Eq. (6) after setting $\rho_+ = 0$ or $\rho_- = 0$, depending on which side the “vacuum” is. Note that the fluid density has canceled out and Eqs. (6) and (7) must be obeyed by all the modes. However, a particular mode, which we have called “trivial” or “universal,” occurs at a free surface for all density profiles. This is the mode $W = e^{-k|y|}$, $\Gamma^2 = \pm k$, the sign being determined by which side the vacuum is (see Ref. [7]). Under inversion a free (fixed) boundary changes into a fixed (free) boundary, and the inversion theorem states that the spectrum remains the same except for the trivial modes which are always present at free boundaries and always absent at fixed boundaries.

Turning to QM we integrate Eq. (2) over an infinitesimal distance, say $0_- < y < 0_+$, and obtain

$$\Delta(D\Psi) = \int_{0_-}^{0_+} V\Psi dy = \Psi(0) \int_{0_-}^{0_+} V(y) dy, \quad (8)$$

where continuity of Ψ has been used to pull it out of the integral sign, just as continuity of W was used to write $\Delta(W\rho) = W\Delta(\rho)$ in Eq. (6). Now, for most potentials the right-hand side (rhs) of Eq. (8) vanishes because the potential remains finite, and we conclude from Eq. (8) that $D\Psi$ is also continuous. However, in the few cases where the rhs is nonzero then $D\Psi$ need not be continuous. The best example is the δ -function potential, $V(y) \sim \delta(y)$, the last example treated in the next section.

To sum up: Ψ and W must be continuous for purely physical reasons. Then Eq. (4a) implies that q also must be continuous. Equation (6) then implies that DW must be continuous. Dq need not be continuous, and $D\Psi$ also need not be continuous [unless the rhs of Eq. (8) vanishes]. Note that there is internal consistency: From Eq. (4a) we have $D\Psi = WDq + qDW$ and therefore $\Delta(D\Psi) = \Delta(WDq) + \Delta(qDW) = W\Delta(Dq)$; hence $\Delta(D\Psi) \neq 0$ unless $\Delta(Dq) = 0$ also.

Our notation, using y to denote the space coordinate, may suggest that only one-dimensional (1D) QM problems are amenable to such correspondence. This need not be the case because the radial Schrödinger equation in spherical geometry reads exactly like Eq. (2) with y denoting the radius r and $V(y) \rightarrow l(l+1)/r^2 + V(r)$, where l is the usual quantum number for angular momentum. One of the examples in the next section will indeed draw upon the radial Schrödinger equation.

III. EXAMPLES

Example 1

The best known example is the exponential density profile, treated first by Lord Rayleigh [1,2]. It may come as no surprise that it corresponds to the QM problem of a "particle in a box."

Consider the profile $\rho = e^{\beta y}$, i.e., $q = e^{\beta y/2}$ in the region $0 \leq y \leq d$, with fixed boundaries or "walls" at $y=0$ and $y=d$. Then

$$V = \frac{\beta^2}{4} - \beta \frac{k^2}{\Gamma^2} = \text{const.} \quad (9)$$

The well-known solution of a particle in a box with zero potential is

$$E_n = \frac{\pi^2 n^2}{d^2}, \quad n = 1, 2, 3, \dots \quad (V=0). \quad (10)$$

Therefore

$$E_n = V + \frac{\pi^2 n^2}{d^2}, \quad V = \text{const.} \quad (11)$$

Combining Eqs. (9), (11), and (4c) we obtain

$$\Gamma^2 = \frac{\beta k^2}{\beta^2/4 + k^2 + \pi^2 n^2/d^2}, \quad (12)$$

which is indeed the growth rate for the RT problem [1,2].

Practically all known analytic solutions of the FD problem involve exponential density profiles which can be understood via Eq. (9): They correspond to constant potentials, the simplest QM problem.

Example 2

The second example involves the "harmonic oscillator," $V = \frac{1}{2} \kappa x^2$. (We use κ , instead of the usual k for the spring constant, so as not to confuse with wave number k .) Already we find a solution to an FD problem which, as far as we know, has not been considered previously.

Let $\rho = e^{-\alpha y^2}$ over $-\infty < y < \infty$, $\alpha > 0$; i.e., a Gaussian density profile. (Such profiles are of interest in a related instability, the Richtmyer-Meshkov instability, occurring in gas curtain experiments [9].) Therefore $q = e^{-\alpha y^2/2}$ and

$$V = x^2 - \alpha - k^4/\Gamma^4, \quad (13)$$

where

$$x \equiv \alpha y + \frac{k^2}{\Gamma^2}. \quad (14)$$

Using $D^2 \equiv d^2/dy^2 = \alpha^2 d^2/dx^2$ in the Schrödinger equation, Eq. (2), we obtain

$$\frac{d^2\Psi}{dx^2} + (\varepsilon - \frac{1}{2} \kappa x^2) \Psi = 0, \quad (15)$$

where

$$\kappa \equiv 2/\alpha^2 \quad (16a)$$

and

$$\varepsilon \equiv \frac{k^4}{\alpha^2 \Gamma^4} + \frac{1}{\alpha} - k^2/\alpha^2. \quad (16b)$$

The following results (and notation) are standard in the harmonic oscillator problem:

$$\varepsilon_n = (n + \frac{1}{2}) \hbar \omega, \quad n = 0, 1, 2, \dots, \quad (17)$$

$$\kappa \equiv m \omega^2, \quad (18)$$

where m is the mass of the particle. Remembering that in our units $\hbar^2 = 2m$, we have

$$\varepsilon_n = (2n + 1)/\alpha, \quad (19)$$

which, combined with Eq. (16b), gives

$$\Gamma^2 = \frac{\pm k}{\sqrt{1 + 2n\alpha/k^2}}, \quad n = 0, 1, 2, \dots \quad (20)$$

Note that for every growing mode there is an oscillating mode. The largest growth or oscillation rate is given by $\Gamma^2 = \pm k$ for $n=0$. A general proof that $\Gamma^2 = \pm k$ is the largest

growth rate in the FD problem was given recently [10]. Note also that $\Gamma^2 = \pm k$ are the two "trivial" modes which are present because the fluid approaches a free boundary having $\rho=0$ as $y \rightarrow \infty$ ($y \rightarrow -\infty$) which is stable (unstable). Of course $n=0$, $\varepsilon_0 = \frac{1}{2}\hbar\omega$, corresponds to the ground state of the harmonic oscillator.

Example 3

Our next example has actually been considered [11] in the past in the context of the inversion theorem without, of course, making the QM connection. (A general treatment of the inversion theorem is given in Kull's review paper [12].) Here we show that it is the Coulomb potential in disguise.

Using a suggestive notation we consider

$$\rho = r^{2(l+1)} \quad \text{or} \quad q = r^{l+1}, \quad 0 \leq r < \infty. \quad (21)$$

Here r stands for y , the space coordinate, and l is any real number, *not necessarily an integer*, satisfying $l+1 > 0$.

Substituting Eq. (21) in Eq. (4b) we have

$$V = \frac{l(l+1)}{r^2} - \frac{Ze^2}{r}, \quad (22)$$

where

$$Ze^2 \equiv \frac{2k^2(l+1)}{\Gamma^2}. \quad (23)$$

The bound-state energies of the Coulomb potential are well-known and usually written in the form

$$E_n = -\frac{1}{2}mc^2\alpha^2 Z^2/n^2, \quad (24a)$$

where c is the speed of light and α is the fine-structure constant $e^2/\hbar c$. In units of $\hbar^2/2m$,

$$E_n = -\frac{Z^2 e^4}{4n^2}. \quad (24b)$$

Combining Eqs. (23) and (4c) we have

$$\Gamma^2 = \frac{k(l+1)}{n}. \quad (25)$$

In the QM problem l and n are integers, but this is not necessarily so in the FD problem. The quantization of angular momentum in QM requires l to be a non-negative integer. This requirement is absent in the FD problem where l is just a parameter describing the steepness of the density profile. The solution [13] of the Schrödinger equation does not require l to be an integer nor, for that matter, that n be an integer. It does require, however, that $n-l-1$ be an integer equal to or greater than zero [13]. Defining

$$m \equiv n-l-1, \quad (26)$$

we must have $m=0,1,2,\dots$. In terms of m , Eq. (25) reads

$$\Gamma^2 = \frac{k(l+1)}{l+1+m} = \frac{k}{1 + \frac{m}{l+1}}, \quad m=0,1,2,\dots, \quad (27)$$

which agrees with Inogamov's result [11] after identifying $l+1$ with his $\beta/2(\rho \sim y^\beta)$. Note that the fastest growing mode ($m=0$) is indeed the trivial mode $\Gamma^2=k$.

Example 4

Consider the profile $\rho = (1 - e^{-y/a})^2$, i.e., $q = 1 - e^{-y/a}$, $0 \leq y < \infty$. As in the previous example we let r stand for the position y and, using Eq. (4b), we find

$$V = \frac{-V_0}{e^{r/a} - 1}, \quad (28)$$

where

$$V_0 \equiv \frac{1}{a^2} \left(1 + \frac{2ak^2}{\Gamma^2} \right). \quad (29)$$

The potential defined in Eq. (28) is known as the Hulthén potential. From the solution of the QM problem (see, for example, Ref. [14]) the bound-state energies are given by

$$\sqrt{-E_n} = \frac{a^2 V_0 - n^2}{2na}, \quad n=1,2,\dots \quad (30)$$

Combining Eqs. (29), (30), and (4c) we have

$$\Gamma^2 = \frac{2ak^2}{n^2 + 2nak - 1}, \quad n=1,2,\dots \quad (31)$$

The fastest mode, $n=1$, again has $\Gamma^2=k$ (note the free surface at $y=0$). There is clearly a one-to-one correspondence between the fastest growing mode and the ground-state in the FD and QM problems, respectively.

Example 5

Our last example is of a somewhat different nature because it involves discontinuous $D\Psi$ and DW , while Ψ , W , and q remain continuous. It is a pure surface tension problem, i.e., we take $q = \text{const}$ but assume that at a certain position in space, say y_s , there is surface tension denoted by $T^{(s)}$. It is straightforward to show (see the Appendix) that in this case the potential is

$$V = \frac{k^4 T^{(s)} \delta(y - y_s)}{\rho \gamma^2}, \quad (32)$$

where ρ is the (constant) density of the fluid and γ is the "growth" rate. Actually, γ^2 is negative and perturbations oscillate at the frequency $|\gamma|/2\pi$.

It is well known [15] that a negative δ potential has one and only one bound state with energy E_{bound}

$$E_{\text{bound}} = -k^2 = -\frac{k^8 T^{(s)2}}{4\rho^2 \gamma^4}, \quad (33)$$

from which we obtain

$$\gamma^2 = -\frac{k^3 T^{(s)}}{2\rho}, \quad (34)$$

in agreement with the FD result [set $\rho_1 = \rho_2 = \rho$ in Eq. (51) of Ref. [2], Chap. X].

IV. INVERSION

Let us identify which QM and FD problems do and do not overlap. There are (infinitely) many FD problems with no QM counterpart, and *vice versa*. For example, density jumps are very common in FD experiments. As we have already mentioned, such problems have no QM analog because any discontinuity in ρ combined with continuity in W would make Ψ discontinuous [see Eq. (4a)] and hence unphysical. Another limitation comes from Eq. (4b): Since ρ or, more precisely, its derivatives “define” the potential V , finite-thickness FD problems [meaning $\rho(y)$ defined over a finite y interval] cannot, in general, be put into correspondence with QM problems. There is no difficulty when the boundaries of the FD problem are fixed, i.e., the fluid is confined within two rigid walls, as was the case in example 1: Such boundaries can be represented by $\rho = \infty$ or $V = \infty$, i.e., a particle confined within the same finite y interval. However, a large class of FD problems have a free boundary and, although one may define such boundaries by $\rho = 0$ and perhaps $V = 0$, this is not sufficient. The difficulty lies in defining Ψ outside this boundary, and it clearly highlights the difference between the physical FD and QM problems: In FD ρ and W are intimately associated, $W(y)$ being the perturbed velocity in the fluid of density $\rho(y)$, and they both exist (or do not exist) over the *same* y interval. In contrast, $\Psi(y)$ exists even where $V(y) = 0$, reminding us that the particle wave function and the potential are not as intimately associated.

Conversely, there is a whole class of QM problems with no FD counterpart: scattering problems. As is well known, these are the solutions of the Schrödinger equation with $E > 0$ or, using Eq. (4c), k imaginary. As far as we know there has been no attempt to give a physical meaning to, let alone solve, the Rayleigh equation with imaginary k , though it may be possible to do so. The Coulomb potential is a good example: It has both bound state ($E < 0$) and scattering ($E > 0$) solutions, and only the former corresponds to a FD problem, as was shown by example 3 in the previous section.

In summary, the set of QM problems and the set of FD problems intersect in a finite region consisting of continuous density profiles and bound-state solutions. Inside that region the connection is given by Eqs. (4a), (4b), and (4c). Even then there are differences of approach: In QM the potential is given and one finds the bound-state energies E_n , which are quantized. In FD the density profile is given and one finds the growth rates Γ_n for an arbitrary wave number k . From the FD point of view the potential is quantized [see Eq. (4b)] while the energy is an arbitrary number.

The inversion theorem [7] applies to the whole set of FD problems having continuous or discontinuous density profiles. To make the connection with QM we will confine ourselves to continuous profiles. Such profiles have an infinite number of growth rates Γ_n , $n = 0, 1, 2, \dots$, which we refer to as the “spectrum” associated with $\rho(y)$. The inversion theorem states that the spectrum is invariant under inversion, defined as $\rho(y) \rightarrow 1/\rho(-y)$:

$$\{\Gamma_0, \Gamma_1, \Gamma_2, \dots\}_{\rho(y)} = \{\Gamma_0, \Gamma_1, \Gamma_2, \dots\}_{1/\rho(-y)} \quad (35)$$

for all k . A general proof for the continuous case was given recently by Kull [12] and we will not go into details, except to point out that under inversion a free ($\rho = 0$) boundary goes over to a fixed ($\rho = \infty$) boundary, and vice versa. Free boundaries support the “trivial” modes $\Gamma^2 = \pm k$ while fixed boundaries do not. When such boundaries are present one must add or subtract the trivial modes from Eq. (35).

Since $\rho(y)$ and $1/\rho(-y)$, two completely different profiles, yield the same spectrum of growth rates, it follows that the spectrum alone cannot determine the density profile, a situation reminiscent of QM. At present it is not known whether one faces a simple dichotomy or a more complicated problem. Our conjecture [7] is that there are no other symmetries, implying that $1/\rho(-y)$ is the only profile having the same spectrum as $\rho(y)$. In this context it is important to note that the spectra are identical for all k and, of course, the density profiles are independent of k [16]. Finally, we emphasize that the *eigenvalues* (Γ_n) are the same, but not the *eigenfunctions* (W_n).

Even if our conjecture is correct and the measurement of Γ_n narrows the field down to two possibilities only, we perceive no practical applications of inversion symmetry in FD because measuring growth rates experimentally is more difficult than measuring density profiles. The opposite is true in QM where measuring phase shifts or bound-state energies is the only way to measure the potential. This is the famous “inverse problem” in QM to which much effort has been dedicated over the past 65 years [8]. Although it was Lord Rayleigh who first posed the question of deducing the density distribution of a string from the frequencies of its vibration [8,17], it was the urgency of deducing $V(y)$ in QM, as opposed to $\rho(y)$ in FD, which has motivated the study of the inverse problem.

For practical reasons most efforts have focused on determining $V(y)$ via the scattering phase shifts and it is well-known that they do not determine $V(y)$ uniquely—there are phase-equivalent potentials [8]. Data about bound states can be used to narrow the choice of the potential [8,18]. However, nonuniqueness occurs even in purely attractive potentials which have only bound states. In fact the best and perhaps the earliest example was provided by Pöschl and Teller [6]. The potential is

$$V(y) = \alpha^2 \left[\frac{\nu(\nu-1)}{\sin^2 \alpha y} + \frac{\mu(\mu-1)}{\cos^2 \alpha y} \right], \quad 0 \leq \alpha y \leq \pi/2, \quad (36)$$

and the spectrum is found to be [6] (see also [14])

$$E_n = \alpha^2 (\mu + \nu + 2n)^2, \quad n = 0, 1, 2, \dots \quad (37)$$

Since E_n depends only on the sum of μ and ν and not on their individual values, it is clear that there are infinitely many potentials having the same spectrum, a point well illustrated by Pöschl and Teller (Fig. 2 in Ref. [6]). For example, the potential given by $\mu = \nu = 2$ has the same spectrum as the substantially different potential given by $\mu = 1.1$, $\nu = 2.9$.

We could not find an analytic expression for the density profile $\rho(y)$ or, equivalently, $q(y)$ corresponding to the above potential. In theory $q(y)$ could be found by solving Eq. (5). In practice it is hard to find analytic solutions to Eq. (5), and therefore we turn to consider some of the examples

given in the preceding section where we identified $q(y)$ and the corresponding $V(y)$. Considering $V(y)$ as a functional of $q(y)$ the inversion theorem states that $V[q(y)]$ and $V[1/q(-y)]$ are “energy-equivalent” potentials. It may be possible to define the Rayleigh equation for imaginary k in which case the inversion theorem could perhaps be extended to phase-equivalent potentials.

The first example is straightforward because an exponential density profile remains exponential under inversion. The boundary conditions, however, change as discussed earlier: The two fixed boundaries go over to free boundaries. It follows that the spectrum is the same as before, Eq. (12), plus the two trivial modes $\Gamma^2 = -k$ and $\Gamma^2 = k$. These modes are also called universal because they occur for *any* density profile $\rho(y)$ between two free boundaries. We should note, however, that going from the fixed-fixed case to the free-free case the eigenfunctions $W_n(y)$ change substantially (they are no longer required to vanish at the boundaries) while their eigenvalues indeed remain the same. Other boundary conditions were considered in Ref. [7].

We now invert example 2, $\rho(y) = e^{-\alpha y^2}$, by letting $\rho(y) \rightarrow 1/\rho(-y) = e^{\alpha y^2}$. This is, of course, quite a different density profile, going to infinity as $y \rightarrow \pm\infty$. The associated QM problem is still given by Eq. (15) after letting $\alpha \rightarrow -\alpha$ in Eqs. (13) and (14). Equation (16a) is unchanged ($\kappa = 2/\alpha^2$) while Eq. (16b) becomes

$$\varepsilon \equiv \frac{k^4}{\alpha^2 \Gamma^4} - \frac{1}{\alpha} - k^2/\alpha^2. \tag{38}$$

Equation (19) remains unchanged because it follows from Eq. (17), remembering that $\alpha > 0$: $\hbar\omega = \sqrt{2m\omega} = \sqrt{2m\sqrt{\kappa/m}} = \sqrt{2\kappa} = \sqrt{4/\alpha^2} = 2/\alpha$. Therefore

$$\varepsilon_n = (2n + 1)/\alpha, \quad n = 0, 1, 2, \dots, \tag{39}$$

as before. Equating Eqs. (38) and (39) we now have

$$\Gamma^2 = \frac{\pm k}{\sqrt{1 + 2(n + 1)\alpha/k^2}}, \quad n = 0, 1, 2, \dots, \tag{40}$$

to be compared with Eq. (20). Better yet, writing Eq. (40) in the form of Eq. (20),

$$\Gamma^2 = \frac{\pm k}{\sqrt{1 + 2n\alpha/k^2}}, \quad n = 1, 2, \dots, \tag{41}$$

we see that the “trivial modes” corresponding to $n = 0$ are automatically eliminated in this inverted problem. The reason, of course, is that the two “free” boundaries at $\pm\infty$ have become fixed in the inverted profile. The fastest growing mode is now given by $k/\sqrt{1 + 2\alpha/k^2}$ instead of k . The difference vanishes for large k , meaning short wavelengths, because these modes are localized near the center and do not “care” about the boundaries. Except for the two trivial modes the rest of the spectrum is identical to the previous one, Eq. (20).

We now invert example 3 given by Eq. (21). The free boundary ($\rho = 0$ for $y \leq 0$) becomes a fixed boundary ($\rho = \infty$ for $y \geq 0$) while for $y < 0$ the profile is given by $1/(-y)^{2(l+1)}$, i.e., $q \sim y^{-(l+1)}$. From Eq. (4b) the potential is found to be

$$V = \frac{(l+1)(l+2)}{r^2} - 2 \frac{k^2(l+1)}{\Gamma^2 r}, \tag{42}$$

where we have identified r with $-y$, so that $0 \leq r < \infty$ as in the QM problem. Defining

$$L \equiv l + 1 \tag{43a}$$

and

$$Ze^2 \equiv \frac{2k^2L}{\Gamma^2} \tag{43b}$$

we have

$$V = \frac{L(L+1)}{r^2} - \frac{Ze^2}{r}, \tag{44}$$

which is identical to the original problem except that the “angular momentum” here is one unit larger than before. However, it is well known that the Coulomb potential has an “accidental degeneracy,” meaning that the energies are independent of the angular momentum, and therefore Eqs. (24) and (25) are valid here also. The only place where the angular momentum enters is in the requirement that $n - L - 1$ be an integer equal to or greater than zero. Hence Eq. (26) becomes

$$m = n - L - 1, \quad m = 0, 1, 2, \dots, \tag{45}$$

or, equivalently, $n = l + 2 + m$. Substituting this value in Eq. (25) we have

$$\Gamma^2 = \frac{k(l+1)}{l+2+m} = \frac{k}{1 + \frac{m+1}{l+1}}, \quad m = 0, 1, 2, \dots, \tag{46}$$

which can be cast in the form of Eq. (27),

$$\Gamma^2 = \frac{k}{1 + \frac{m}{l+1}}, \quad m = 1, 2, \dots \tag{47}$$

This form exhibits explicitly that the spectrum is the same as before except that the trivial mode $\Gamma^2 = k$ ($m = 0$) is not present, as noted by Inogamov [11].

In contrast to the preceding example [compare Eqs. (42) and (47)] even short wavelength perturbations ($k \rightarrow \infty$) do not approach $\Gamma^2 = k$. The reason probably lies in the fact that the “harmonic oscillator” problem had a length scale with which wavelengths could be compared, while the “Coulomb” problem has no such scale.

Next we invert the profile of example 4 defined by $q = 1 - e^{-y/a}$, $y \geq 0$. The free surface at $y = 0$ goes over to a fixed surface, i.e., $\rho = \infty$ for $y \geq 0$ in the inverted profile, and $q = 1/(1 - e^{y/a})$ for $y \leq 0$. From the inversion theorem we know that the spectrum is still given by Eq. (31) with the $n = 1$ mode missing, so that the fastest growing mode is now given by $\Gamma^2 = k/(2 + 3/2ak)$. The associated potential has a somewhat complicated form (an Eckart potential [19]) and need not be given here.

We end with our last example, example 5, which is the simplest of all because $\rho(y)$ is constant: Letting $\rho \rightarrow 1/\rho$ merely changes the value of that constant and since the Rayleigh equation happens to be linear in $\rho(y)$ a change in scale is immaterial. The inversion theorem relies heavily on this property of the Rayleigh equation without which inversion would not make sense dimensionwise. This linearity in $\rho(y)$ must not be confused with the "linear approximation" mentioned in the Introduction which makes the Rayleigh equation linear in $W(y)$. Just as the Rayleigh equation is unaffected if we multiply $\rho(y)$ by a constant, the associated potential defined in Eq. (4b) is also unaffected.

V. CONCLUSIONS

A practical consequence of our analysis is the following: Before solving the Rayleigh equation for a given density profile construct the potential and see if the corresponding Schrödinger equation has already been solved. This is usually the case. If not, try the inverted profile: The Schrödinger equation for $V[1/\rho(-y)]$ may have been solved in which case the inversion theorem can be used to deduce the spectrum of $\rho(y)$, adding or subtracting the trivial modes as appropriate. Of course if neither $V(y)$ nor $V[1/\rho(-y)]$ has been solved then this approach is useless.

To go the other way, i.e., construct $\rho(y)$ from $V(y)$, requires solving Eq. (5), a nontrivial task. As mentioned in the preceding section, it would be interesting to find $\rho(y)$ associated with the $V(y)$ in Eq. (36). This problem is similar to example 1, "particle in a box," so it has probably an exponential form with fixed surfaces at $\alpha y = 0$ and $\pi/2$. A reasonable ansatz which does not work is $q = \sin^\nu \alpha y \cos^\mu \alpha y$.

Despite their similarities there are several differences between the QM and the FD problems. As we mentioned, one can define Ψ where $V=0$, but cannot define W where $\rho=0$. QM has both bound and scattering solutions, while FD has only "bound" states. Even if we define $\rho = \text{const}$ as a fluid boundary representing a region where $V=0$ (since $Dq = D^2q = 0$ for constant q), the FD eigenfunctions must decay as $W(y) \sim e^{-k|y|}$ in these regions while $\Psi(y) \sim e^{\pm iky}$ for QM.

We have focused on the pure RT problem because it arises in inertial-confinement fusion [5]. The pure KH (Kelvin-Helmholtz) problem has also a simple potential,

$$V_{\text{KH}} = \frac{D^2 U}{U - c}, \quad (48)$$

where $U(y)$ is the velocity profile and c is a phase speed, so that a connection with the Schrödinger equation can again be established. The potential for the more general coupled RT/KH problem is more complicated and given in the Appendix.

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APPENDIX

In this appendix we discuss how the potential is affected by the presence of shear flow in the x direction and by surface tension at interfaces (x - z planes) located at y_i . The FD problem is discussed in Chap. XI of [2], and the general equation appears as Eq. (16) in that chapter, which is a generalization of Eq. (1) in this paper. It is straightforward to show that the potential for the corresponding Schrödinger equation is given by

$$V(y) = \frac{D^2 q}{q} + \frac{k_x D^2 U + 2k_x (DU)(Dq)/q}{Uk_x - i\gamma} + \frac{2gk^2(Dq)/q - \frac{k^4}{q^2} \sum_i T_i^{(s)} \delta(y - y_i)}{(Uk_x - i\gamma)^2}. \quad (A1)$$

In this expression $T_i^{(s)}$ is the surface tension of the interface located at y_i , and $U(y)$ is the velocity profile. As before, $q(y) = \sqrt{\rho(y)}$ and γ is the growth rate, i.e., perturbations η evolve as $e^{\gamma t}$. One difference with the pure RT case is that γ can now be complex, and $c \equiv i\gamma/k_x$ is usually called the phase velocity so that $\eta \sim e^{ik_x(x-ct)}$.

Equation (4b) is recovered for $U = \text{const}$ and hence, by Galilean invariance, $U=0$. The pure KH potential is given by $q = \text{const}$, so that $V_{\text{KH}} = D^2 U / (U - c)$ as given in Eq. (48). If both U and q are constant then one has a pure surface tension problem with the potential given by Eq. (32).

Since c is generally complex, so is V . Complex potentials are associated with the creation or absorption of particles. However, problems of "neutral stability," $c=0$, have been extensively studied in FD and therefore we can associate such problems with real potentials which are, of course, more familiar. We illustrate with an example. Consider a modified Pöschl-Teller potential,

$$V = \frac{-V_0}{\cosh^2 \alpha y}, \quad (A2)$$

which has bound states given by [14]

$$E = -\frac{\alpha^2}{4} (-1 - 2n + \sqrt{1 + 4V_0/\alpha^2})^2 \quad (A3)$$

where n is a non-negative integer satisfying $2n < \sqrt{1 + 4V_0/\alpha^2} - 1$. The corresponding FD problem is the well-known hyperbolic tangent profile [2],

$$U = U_0 \tanh \alpha y, \quad (A4)$$

so that

$$V_{\text{KH}} = \frac{-2\alpha^2}{(1 - c/U) \cosh^2 \alpha y}. \quad (A5)$$

The wave number corresponding to neutral stability can be obtained by setting $c=0$. Then the problem reduces to Eq. (A2) with $V_0 = 2\alpha^2$. Therefore $E = -(\alpha^2/4)[-1 - 2n + 3]^2$

$= -\alpha^2(1-n)^2$, $n < 1$, and we find ($n=0$ since it must be non-negative) $k^2 = -E = \alpha^2$ which is indeed the wave number having $c=0$. (See pp. 494–498 in Ref. [2]. Note that we have assumed no density stratification, i.e., a pure KH

problem, usually described as “zero Richardson number.”) The associated eigenfunctions can be found using Eq. (4a), $W \sim \Psi$ (since $\rho = \text{const}$) and is expressed in terms of hypergeometric functions [14].

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